# ON THE EXISTENCE OF THE UNIVERSAL CLASSES FOR ALGEBRAIC GROUPS

ANTOINE TOUZÉ

ABSTRACT. In this note, we apply the ideas developped by M. Chałupnik in [C] to the framework of strict polynomial bifunctors. This allows us to get a new proof of the existence of the 'universal classes' originally constructed in [T1].

## 1. INTRODUCTION

Let $\Bbbk$ be a field of prime characteristic $p$, and let $GL_{n,\Bbbk}$ denote the general linear group scheme over $\Bbbk$. In [T1], we exhibited a set of 'universal classes' $c[i]_{i\in\mathbb{N}}$ living in the cohomology of $GL_{n,\Bbbk}$. These classes' existence was anticipated by van der Kallen [VdK], and they are one of the key ingredients to prove van der Kallen's conjecture, which is now a theorem:

**Theorem** ([TVdK]). *Let $G$ be a reductive algebraic group scheme over a field $\Bbbk$, and let $A$ be an finitely generated $\Bbbk$-algebra acted on by $G$ via algebra automorphisms. Then the cohomology $H^*(G,A)$ is finitely generated as a graded $\Bbbk$-algebra.*

The purpose of this note is to give a new proof of the existence of the universal classes $c[i]_{i\in\mathbb{N}}$. To be more specific, if $V$ is a finite dimensional vector space and $d\geq 1$, the vector space $V^{\otimes d}$ is acted on by the symmetric group $\mathfrak{S}_d$. We denote by $\Gamma^d(V)$ the '$d$-th divided power of $V$', that is the subspace of invariants $(V^{\otimes d})^{\mathfrak{S}_d}$. We also denote by $\mathfrak{gl}_n$ the adjoint representation of $GL_{n,\Bbbk}$ and by $\mathfrak{gl}_n^{(1)}$ the representation obtained by base change along the Frobenius twist. We give in section 4 a new proof of the following theorem, originally established in [T1, Thm 0.1].

**Theorem 1.1.** *Let $\Bbbk$ be a field of positive characteristic and let $n\geq p$ be an integer. There are cohomology classes $c[d]\in H^{2d}(GL_{n,\Bbbk},\Gamma^d(\mathfrak{gl}_n^{(1)}))$ such that :*

1. *$c[1]\in H^2(GL_{n,\Bbbk},\mathfrak{gl}_n^{(1)})$ is non zero.*
2. *If $d\geq 1$ and $\Delta_{(1,\dots,1)}:\Gamma^d(\mathfrak{gl}_n^{(1)})\to(\mathfrak{gl}_n^{(1)})^{\otimes d}$ is the inclusion, then $\Delta_{(1,\dots,1)\,*}c[d]=c[1]^{\cup d}$.*

In the 'old proof' we built the universal classes by computing explicit cycles, using explicit coresolutions of the representation $\Gamma^d(\mathfrak{gl}_n^{(1)})$. To achieve this construction, we used two main ingredients: the twist compatible category, constructed in [T1, Section 3], and a result on the combinatorics of tensor products of $p$-complexes [T1, Prop 2.4].

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The new proof uses the ideas of the article [C] and involves rather different ingredients. It relies heavily on derived categories, a formality phenomenon discovered in [T3] and the adjunction argument used by M. Chałupnik to prove the collapsing conjecture [T3, Conjecture 8.1] (suitably adapted to the world of strict polynomial bifunctors).

As a common point, the two proofs rely on the very fundamental complexes constructed by Troesch in [Tr] (see also [T3, Section 9] for a slightly different presentation of these complexes).

## 2. Functors and bifunctors

Our proof of theorem 1.1 uses the category of strict polynomial bifunctors introduced in [FF]. In this section, we recall the main facts that we will need about this category. As a reference about functors, bifunctors and other generalizations, we refer the reader to [FS, Section 2], [SFB, Section 3], [FF, Section 1], [P, Section 4], [F, Section 3], [T2, Section 2], or [T4, Section 2].

### 2.1. **Functors.**

Let us first begin with brief recollections of the simpler category of strict polynomial functors introduced by Friedlander and Suslin in [FS]. If $\Bbbk$ is a field of prime characteristic $p$, we denote by $\mathcal{P}_d$ the abelian category of homogeneous strict polynomial functors of degree $d$ over $\Bbbk$. The objects of $\mathcal{P}_d$ are nice endofunctors of the category $\mathcal{V}_\Bbbk$ of finite dimensional $\Bbbk$-vector spaces, which naturally arise in representation theory of algebraic groups, and the morphisms of $\mathcal{P}_d$ are some natural transformations between these functors.

Examples of objects of $\mathcal{P}_d$ include the $d$-th tensor power $\otimes^d : V \mapsto V^{\otimes d}$, the $d$-th symmetric power $S^d : V \mapsto S^d(V)$, and the $d$-th divided power $V \mapsto \Gamma^d(V) = (V^{\otimes^d})^{\mathfrak{S}_d}$. The $r$-th Frobenius twist $I^{(r)} \in \mathcal{P}_{p^r}$ is the subfunctor of $S^{p^r}$ such that $I^{(r)}(V)$ is generated by the elements of the form $v^{p^r} \in S^{p^r}(V)$. As usual, if $F \in \mathcal{P}_d$, we denote by $F^{(r)}$ the composition $F \circ I^{(r)}$.

We denote by $F^\sharp$ the dual of a functor $F$, that is $F^\sharp(V) := F(V^\vee)^\vee$, where the symbol '$^\vee$' stands for $\Bbbk$-linear duality. We have

$$\mathrm{Hom}_{\mathcal{P}_d}(F, G) = \mathrm{Hom}_{\mathcal{P}_d}(G^\sharp, F^\sharp)\ .$$

If $F \in \mathcal{P}_d$, and $X \in \mathcal{V}_\Bbbk$ we let $F^X$ and $F_X$ be the functors with parameter $X$:

$$F^X : V \mapsto F(\mathrm{Hom}_\Bbbk(X, V))\ , \quad F^X : V \mapsto F(X \otimes V)\ .$$

The notation $F^X$ reminds that $F^X(V)$ is contravariant with respect to $X$ (compare the usual notation for functional spaces), while $F_X(V)$ is covariant with respect to $X$. The $S^d_X$, $X \in \mathcal{V}_\Bbbk$ form an injective cogenerator of $\mathcal{P}_d$ while the $\Gamma^{d,X} := (\Gamma^d)^X = (S^d_X)^\sharp$ form a projective generator of $\mathcal{P}_d$. We recall the isomorphisms, natural in $F, X$:

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,X}, F) \simeq F(X)\ , \quad \mathrm{Hom}_{\mathcal{P}_d}(F, S^d_X) \simeq F^\sharp(X)\ .$$

These isomorphisms are nothing but a disguised form of the Yoneda lemma (cf. [P, Section 4], or [T4, Section 2]) so we simply call them 'the Yoneda isomorphisms'.

2.2. **Bifunctors.** If $F \in \mathcal{P}_d$, then the vector space $F(\Bbbk^n)$ is canonically endowed with an action of the group scheme $GL_{n,\Bbbk}$, and Friedlander and Suslin proved [FS, Cor 3.13] that the evaluation map

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F,G) \to \mathrm{Ext}^*_{GL_{n,\Bbbk}}(F(\Bbbk^n),G(\Bbbk^n)) = H^*\big(GL_{n,\Bbbk}, \mathrm{Hom}_{\Bbbk}(F(\Bbbk^n),G(\Bbbk^n))\big)$$

is an isomorphism if $n \geq d$. (This allows to perform Ext-computations in $\mathcal{P}_d$, where computations are surprisingly easier). Strict polynomial bifunctors were used in [FF] to generalize this formula to more general $GL_{n,\Bbbk}$-representations than the ones of the somewhat restrictive form $\mathrm{Hom}_{\Bbbk}(F(\Bbbk^n),G(\Bbbk^n))$. If $d,e \geq 0$, we denote by $\mathcal{P}_e^d$ the abelian category of strict polynomial bifunctors which are homogeneous of bidegree $(d,e)$, contravariant with respect to their first variable and covariant with respect to the second variable.

Thus, objects of $\mathcal{P}_e^d$ are nice bifunctors $B : (V,W) \mapsto B(V,W)$, with $V,W \in \mathcal{V}_\Bbbk$, contravariant in $V$ and covariant in $W$ and taking values in $\mathcal{V}_\Bbbk$. The vector spaces $B(\Bbbk^n,\Bbbk^m)$ are canonically endowed with a left action of $GL_{m,\Bbbk}$ and a right action of $GL_{n,\Bbbk}$ which commute. Using the inverse in $GL_{n,\Bbbk}$, and taking $n = m$ and the diagonal action, we get an action of $GL_{n,\Bbbk}$ on $B(\Bbbk^n,\Bbbk^n)$. For example, let $gl \in \mathcal{P}_1^1$ denote the bifunctor $(V,W) \mapsto \mathrm{Hom}_\Bbbk(V,W)$. Then $\mathrm{gl}(\Bbbk^n,\Bbbk^n)$ is nothing but the adjoint representation $\mathfrak{gl}_n$ of $GL_{n,\Bbbk}$. There are several ways to construct bifunctors from functors. If $F \in \mathcal{P}_d$ and $G \in \mathcal{P}_e$, we denote by $\mathcal{H}om(F,G) \in \mathcal{P}_e^d$ the bifunctor:

$$\mathcal{H}om(F,G) : (V,W) \mapsto \mathrm{Hom}_\Bbbk(F(V),G(W)) \ .$$

We also denote by $F\mathrm{gl} \in \mathcal{P}_d^d$ the bifunctor:

$$F\mathrm{gl} : (V,W) \mapsto F(\mathrm{gl}(V,W)) \ .$$

Franjou and Friedlander proved [FF, Thm 1.5] the following generalization of Friedlander and Suslin's isomorphism. For all $n \geq 1$ there is a map

$$\mathrm{Ext}^*_{\mathcal{P}_d^d}(\Gamma^d\mathrm{gl}, B) \to H^*(GL_{n,\Bbbk}, B(\Bbbk^n,\Bbbk^n)) \ ,$$

and this map is an isomorphism if $n \geq d$. For this reason, the extensions on the left hand side are called the 'bifunctor cohomology of $B$' and written under the more sugestive notation $H^*_{\mathcal{P}}(B)$.

Let us recall some basic facts about the structure of $\mathcal{P}_e^d$. We define the dual $B^\sharp$ of a bifunctor by letting $B^\sharp(V,W) := B(V^\vee,W^\vee)^\vee$, so that we have

$$\mathrm{Hom}_{\mathcal{P}_e^d}(B,C) = \mathrm{Hom}_{\mathcal{P}_e^d}(C^\sharp,B^\sharp) \ .$$

The functors $\mathcal{H}om(\Gamma^d_X, S^e_Y)$, $X,Y \in \mathcal{V}_\Bbbk$ form an injective cogenerator of $\mathcal{P}_e^d$ and their duals $\mathcal{H}om(S^{d,X},\Gamma^{e,Y})$, $X,Y \in \mathcal{V}_\Bbbk$ form a projective generator. There are isomorphisms, natural in $B,X,Y$:

$$\mathrm{Hom}_{\mathcal{P}_e^d}(\mathcal{H}om(S^{d,X},\Gamma^{e,Y}),B) \simeq B(X,Y) \ ,$$
$$\mathrm{Hom}_{\mathcal{P}_e^d}(B,\mathcal{H}om(\Gamma^d_X,S^e_Y)) \simeq B^\sharp(X,Y) \ .$$

We shall call these isomorphisms 'the Yoneda isomorphisms', as in the case of ordinary functors. Finally, we recall two basic formulas relating morphisms

in $\mathcal{P}_e^d$ to morphisms in $\mathcal{P}_d$ and $\mathcal{P}_e$, namely:

$$\mathrm{Hom}_{\mathcal{P}_e^d}(\mathcal{H}om(E,F),\mathcal{H}om(G,H)) \simeq \mathrm{Hom}_{\mathcal{P}_d}(G,E)\otimes \mathrm{Hom}_{\mathcal{P}_e}(F,H)\ ,$$
$$\mathrm{Hom}_{\mathcal{P}_e^e}(\Gamma^d\mathrm{gl},\mathcal{H}om(F,G)) \simeq \mathrm{Hom}_{\mathcal{P}_d}(F,G)\ .$$

## 3. The cohomology of twisted bifunctors

In this section, we study the cohomology of twisted bifunctors. Our approach follows the ideas of [C].

3.1. **The adjunction argument.** In this paragraph, we adapt the adjunction argument of [C, Section 2] to the category of bifunctors. If $B \in \mathcal{P}_e^d$, we denote by $B^{(r)}$ its precomposition by the $r$-th Frobenius twist on both variables:

$$B^{(r)} : (V,W) \mapsto B(V^{(r)},W^{(r)})\ .$$

So precomposition by the Frobenius twist induce an exact functor:

$$\mathrm{Tw}_r : \mathcal{P}_e^d \to \mathcal{P}_{p^r e}^{p^r d}\ .$$

The following formula gives an explicit expression for its left adjoint $\ell_r$. We define $\ell_r(B)$ to be the *dual* of the bifunctor:

$$(V,W) \mapsto \mathrm{Hom}_{\mathcal{P}_{ep^r}^{dp^r}}(B,\mathcal{H}om(\Gamma^d_V,S^e_W)^{(r)})\ .$$

**Proposition 3.1.** *The functors $(\ell_r,\mathrm{Tw}_r)$ form an adjoint pair.*

*Proof.* We have to prove an isomorphism, natural in $B$, $B'$:

$$\mathrm{Hom}_{\mathcal{P}^{d_e}}(\ell_r(B),B') \simeq \mathrm{Hom}_{\mathcal{P}_{p^r e}^{p^r d}}(B,(B')^{(r)})\ . \quad (*)$$

Since the bifunctors $\mathrm{Hom}_{\mathcal{P}^{d_e}}(-,-)$ and $\mathrm{Hom}_{\mathcal{P}_{p^r e}^{p^r d}}(-,-)$ are left exact with respect to both variables,it suffices to build isomorphism $(*)$ when $B$ is a projective generator and $B'$ is an injective cogenerator (the general result follows by taking resolutions).

But if $B = \mathcal{H}om(S^{dp^r,V},\Gamma^{ep^r,W})$ and $B' = \mathcal{H}om(\Gamma^d_X,S^e_Y)$, we can identify the right hand side of formula $(*)$ via the Yoneda isomorphism:

$$\mathrm{Hom}_{\mathcal{P}_{ep^r}^{dp^r}}(B,(B')^{(r)}) \simeq S^{d\,(r)}((X\otimes V)^\vee)\otimes S^{e\,(r)}(Y\otimes W)\ . \quad (1)$$

We can also identify the left hand side of the formula via the Yoneda isomorphism:

$$\mathrm{Hom}_{\mathcal{P}^{d_e}}(\ell_r(B),B') \simeq \ell_r(B)^\sharp(X,Y)\ . \quad (2)$$

Finally, we can compute $\ell_r(B)^\sharp(X,Y) = \mathrm{Hom}_{\mathcal{P}_{ep^r}^{dp^r}}(B,\mathcal{H}om(\Gamma^d_X,S^e_Y)^{(r)}))$, once again thanks to a Yoneda isomorphism:

$$\ell_r(B)^\sharp(X,Y) \simeq S^{d\,(r)}((X\otimes V)^\vee)\otimes S^{e\,(r)}(Y\otimes W)\ . \quad (3)$$

Putting the isomorphisms (1), (2) and (3) together, we construct the isomorphism $(*)$, natural in $B,B'$ when $B$ is a projective generator and $B'$ is an injective cogenerator. This concludes the proof. $\square$

Now we work in the bounded above derived category $\mathbf{D}^-\mathcal{P}_e^d$ (see e.g. [W, Chapter 10] or [K]). Since $Tw_r$ is exact, its right derived functor:

$$\mathbf{R}\mathrm{Tw}_r : \mathbf{D}^-\mathcal{P}_e^d \to \mathbf{D}^-\mathcal{P}_{ep^r}^{dp^r}$$

is simply defined by sending an object $C \in \mathbf{D}^+\mathcal{P}_e^d$ (that is, a bounded above complex $C$ of bifunctors) to the complex $C^{(r)} = \mathrm{Tw}_r(C)$. Since $\mathcal{P}_e^d$ has enough projectives, the left derived functor of $\ell_r$ is defined on $\mathbf{D}^-\mathcal{P}_{ep^r}^{dp^r}$

$$\mathbf{L}\ell_r : \mathbf{D}^-\mathcal{P}_{ep^r}^{dp^r} \to \mathbf{D}^-\mathcal{P}_e^d\ .$$

The following lemma is an easy check from the definition of total derived functors.

**Lemma 3.2.** *For all $C \in \mathbf{D}^-\mathcal{P}_{ep^r}^{dp^r}$ we have a natural isomorphism:*

$$\mathbf{L}\ell_r(C)(V,W) \simeq \left(\mathbf{R}\mathrm{Hom}_{\mathcal{P}_{ep^r}^{dp^r}}(C, \mathcal{H}om(\Gamma_V^d, S_W^e)^{(r)}\right)^\sharp$$

The following statement is a formal consequence of proposition 3.1 (adapt the proof of [W, Thm 10.7.6] or see [K, Section 13])

**Proposition 3.3.** *The functors $(\mathbf{L}\ell_r, \mathbf{R}\mathrm{Tw}_r)$ form an adjoint pair.*

3.2. **The formality argument.** Now we use the formality phenomenon discovered in [T3, Section 4] to get an explicit computation (in the derived category) of the complex $\mathbf{L}\ell_r(\Gamma^{dp^r}\mathrm{gl})$.

We first need a few notations. If $B \in \mathcal{P}_e^d$ and $Z \in \mathcal{V}_\Bbbk$, we denote by $B_Z \in \mathcal{P}_e^d$ the bifunctor:

$$B_Z : (V,W) \mapsto B(V, Z\otimes W)\ .$$

If $Z$ is a finite dimensional graded vector space, then the functor $B_Z$ inherits a grading, defined similarly as in [T3, Section 2.5]. To be more specific, let the multiplicative group $\mathbb{G}_m$ act on each $Z^i$ with weight $i$, and trivially (i.e. with weight zero) on $V$ and $W$. Then $B(V, Z\otimes W)$ inherits an action of $\mathbb{G}_m$. By definition, the elements of $B(V, Z\otimes W)$ of degree $j$ are the elements of weight $j$ under this action of $\mathbb{G}_m$.

**Proposition 3.4.** *Let $E_r$ denote the graded vector space with $(E_r)_{2i} = \Bbbk$ if $0 \le i < p^r$ and $(E_r)_j = 0$ otherwise. Consider the graded functor $\Gamma^d(\mathrm{gl})_{E_r}$ as a complex with trivial differential. There is an isomorphism in $\mathbf{D}^-\mathcal{P}_e^d$:*

$$\mathbf{L}\ell_r(\Gamma^{dp^r}\mathrm{gl}) \simeq \Gamma^d\mathrm{gl}_{E_r}\ .$$

*Proof.* It suffices to show the isomorphism $\mathbf{D}^-\mathcal{P}_e^d$ :

$$\mathbf{R}\mathrm{Hom}_{\mathcal{P}_{dp^r}^{dp^r}}(\Gamma^{dp^r}\mathrm{gl}, \mathcal{H}om(\Gamma_V^d, S_W^e)^{(r)})) \simeq \left(\Gamma^d\mathrm{gl}_{E_r}\right)^\sharp(V,W) = S^d\mathrm{gl}_{E_r}(V,W)\ .$$

Let $P$ denote a projective resolution of $\Gamma_V^{d\,(r)}$ and $J$ denote an injective resolution of $S_W^{e\,(r)}$. Then the left hand side is isomorphic to the complex $\mathrm{Hom}_{\mathcal{P}_{dp^r}^{dp^r}}(\Gamma^{dp^r}\mathrm{gl}, \mathcal{H}om(P,J))$. The latter is isomorphic to the complex $\mathrm{Hom}_{\mathcal{P}_{dp^r}}(P,J)$, hence to the complex $\mathrm{Hom}_{\mathcal{P}_{dp^r}}((\Gamma_V^d)^{(r)}, J)$.

Now if we choose for $J$ a direct sum of Troesch complexes, we know from [T3, Lemmas 4.1 and 4.4] an isomorphism in $\mathbf{D}^-\mathcal{P}_e^d$:

$$\mathrm{Hom}_{\mathcal{P}_{dp^r}}((\Gamma_V^d)^{(r)}, J) \simeq \mathrm{Hom}_{\mathcal{P}_d}(\Gamma_V^d, (S_W^d)_{E_r})\ .$$

To conclude the proof, we use the Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d_V, (S^d_W)_{E_r}) \simeq S^d(V^\vee \otimes W \otimes E_r) \simeq S^d\mathrm{gl}_{E_r}(V,W)\ .$$

□

3.3. **Cohomology of twisted bifunctors.** Now we turn to the study of the cohomology of twisted bifunctors, that is, the study of extensions of the form:

$$H^*_{\mathcal{P}}(B^{(r)}) := \mathrm{Ext}^*_{\mathcal{P}^{dp^r}_{dp^r}}(\Gamma^{dp^r}\mathrm{gl}, B^{(r)})\ .$$

We first need a technical lemma on graded bifunctors.

**Lemma 3.5.** *Let $B, B' \in \mathcal{P}^d_e$, and let $Z$ denote a finite dimensional graded vector space, and let $Z^\vee$ denote its dual, graded so that $Z^\vee \simeq Z$. There is an isomorphism of graded vector spaces, natural in $B, B'$:*

$$\mathrm{Hom}_{\mathcal{P}^d_e}(B_Z, B') \simeq \mathrm{Hom}_{\mathcal{P}^d_e}(B, B'_{Z^\vee})\ . \quad (*)$$

*Since the functor $B \mapsto B_Z$ is exact, this isomorphism induces an isomorphism on the level of the derived category:*

$$\mathrm{Hom}_{\mathbf{D}^-\mathcal{P}^d_e}(B_Z, B') \simeq \mathrm{Hom}_{\mathbf{D}^-\mathcal{P}^d_e}(B, B'_{Z^\vee})\ . \quad (*)$$

*Proof.* By left exactness of the bifunctor $\mathrm{Hom}_{\mathcal{P}^d_e}(-,-)$ with respect to both variables, it suffices to build isomorphism $(*)$ when $B$ is a projective generator and $B'$ is an injective cogenerator (the general result follows by taking resolutions).

But if $B = \mathcal{H}om(S^{d,V}, \Gamma^{e,W})$ and $B' = \mathcal{H}om(\Gamma^d_X, S^e_Y)$ we may compose the two Yoneda isomorphisms:

$$\mathrm{Hom}_{\mathcal{P}^d_e}(B_Z, B') \simeq S^d((X\otimes V)^\vee) \otimes S^e(Y \otimes Z^\vee \otimes W)\ ,\ \text{and}$$
$$\mathrm{Hom}_{\mathcal{P}^d_e}(B, B'_{Z^\vee}) \simeq S^d((X\otimes V)^\vee) \otimes S^e(Y \otimes Z^\vee \otimes W)$$

to obtain the result. □

We are now ready to prove the generalization of the collapsing result [C, Cor 3.3] to the framework of bifunctors.

**Theorem 3.6.** *Let $r$ be a positive integer and let $E_r$ denote the graded vector space with $(E_r)_{2i} = \Bbbk$ if $0 \le i < p^r$ and $(E_r)_j = 0$ otherwise. There is an isomorphism of graded vector spaces, natural in $B \in \mathcal{P}^d_d$ (take the total grading on th right hand side):*

$$H^*_{\mathcal{P}}(B^{(r)}) \simeq H^*_{\mathcal{P}}(B_{E_r})\ .$$

*Proof.* Let $C$ be an object of $\mathbf{D}^-\mathcal{P}^d_d$. We have isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^-\mathcal{P}^{dp^r}_{dp^r}}(\Gamma^{dp^r}\mathrm{gl}, C^{(r)}) &\simeq \mathrm{Hom}_{\mathbf{D}^-\mathcal{P}^d_d}(\mathbf{L}\ell(\Gamma^{dp^r}\mathrm{gl}), C)\\ &\simeq \mathrm{Hom}_{\mathbf{D}^-\mathcal{P}^d_d}(\Gamma^d\mathrm{gl}_{E_r}, C)\\ &\simeq \mathrm{Hom}_{\mathbf{D}^-\mathcal{P}^d_d}(\Gamma^d\mathrm{gl}, C_{E_r})\ .\end{aligned}$$

The first isomorphism follows by adjunction (proposition 3.1), the second isomorphism by formality (proposition 3.4) and the last one from lemma 3.5 with the isomorphism $E_r^\vee \simeq E_r$. If $B \in \mathcal{P}^d_d$, we apply this isomorphism to $C = B[i]$ and take homology to get the result. □

## 4. Proof of theorem 1.1

In this section, we prove theorem 1.1. We first transpose the problem in the framework of strict polynomial functors as in [T1, Section 1.2]. Using the map

$$H^*_{\mathcal{P}}(B) \to H^*(GL_{n,\Bbbk}, B(\Bbbk^n, \Bbbk^n))\,,$$

together with the fact that for $n \geq p$, it induces an isomorphism

$$H^2_{\mathcal{P}}(\mathrm{gl}^{(1)}) \xrightarrow{\simeq} H^2(GL_{n,\Bbbk}, \mathfrak{gl}_n^{(1)})\ ,$$

we easily see that theorem 1.1 is implied by the following theorem.

**Theorem 4.1.** *Let $\Bbbk$ be a field of positive characteristic $p$. There are cohomology classes $c[d] \in H^{2d}_{\mathcal{P}}(\Gamma^{d\,(1)}\mathrm{gl})$ satisfying the following conditions.*

1. *$c[1]$ is non zero.*
2. *If $d \geq 1$ and $\Delta_{(1,\dots,1)} : \Gamma^{d\,(1)}\mathrm{gl} \to \otimes^{d\,(1)}\mathrm{gl}$ is the inclusion, then $\Delta_{(1,\dots,1)\,*}c[d] = c[1]^{\cup d}$.*

So we are left with the problem of finding classes $c[d] \in H^{2d}_{\mathcal{P}}(\Gamma^{d\,(1)}\mathrm{gl})$. Finding $c[1]$ is not a problem. Indeed, it is well-known that $H^2_{\mathcal{P}}(I^{(1)}\mathrm{gl}) \simeq \Bbbk \neq 0$ (this results for example from theorem 3.6 but one can find much more elementary proofs of this computation). So we can choose for $c[1]$ a non zero cohomology class in $H^2_{\mathcal{P}}(I^{(1)}\mathrm{gl})$.

Now we want to find the classes $c[d]$ for $d \geq 2$. The action of the symmetric group $\mathfrak{S}_d$ on $\otimes^d$ (by permuting the factors of the tensor product) induce an action on the graded vector space $H^*_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})$.

**Lemma 4.2.** *For all $d \geq 2$ the cup product $c[1]^{\cup d} \in H^*_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})$ is invariant under the action of the symmetric group.*

*Proof.* Recall that the cup product:

$$H^i_{\mathcal{P}}(B) \otimes H^j_{\mathcal{P}}(B') \xrightarrow{\cup} H^{i+j}(B \otimes B')$$

is defined as the composite of the external product of extensions

$$\mathrm{Ext}^i_{\mathcal{P}_d^d}(\Gamma^d\mathrm{gl}, B) \otimes \mathrm{Ext}^j_{\mathcal{P}_e^e}(\Gamma^d\mathrm{gl}, B) \to \mathrm{Ext}^{i+j}_{\mathcal{P}_{d+e}^{d+e}}(\Gamma^d\mathrm{gl} \otimes \Gamma^e\mathrm{gl}, B \otimes B')$$

and the map induced by the comultiplication $\Gamma^{d+e}\mathrm{gl} \to \Gamma^d\mathrm{gl} \otimes \Gamma^e\mathrm{gl}$. Since $c[1]$ is in even degree and $\Gamma^*\mathrm{gl}$ is a cocommutative coalgebra, one easily gets the result from the definition of the action of $\mathfrak{S}_d$ and the definition of the cup product. □

In view of lemma 4.2, it suffices to prove that all the classes of $H^{2d}_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})^{\mathfrak{S}_d}$ are obtained from classes of $H^{2d}_{\mathcal{P}}(\Gamma^{d\,(1)}\mathrm{gl}$ through the map

$$\Delta_{(1,\dots,1)\,*} : H^{2d}_{\mathcal{P}}(\Gamma^{d\,(1)}\mathrm{gl}) \to H^{2d}_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})\ .$$

Actually, we can get a slightly more general statement from theorem 3.6.

**Proposition 4.3.** *Let $d \geq 2$. Then the map $\Delta_{(1,\dots,1)\,*}$ induces a surjection*

$$H^*_{\mathcal{P}}(\Gamma^{d\,(1)}\mathrm{gl}) \twoheadrightarrow H^*_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})^{\mathfrak{S}_d}\ .$$

*Proof.* Let us first remark that $I^{(1)}\mathrm{gl} \simeq \mathrm{gl}^{(1)}$, so theorem 3.6 yields a commutative diagram:

$$\begin{tikzcd} H^*_{\mathcal{P}}(\Gamma^{d\,(1)}\mathrm{gl}) \arrow[r, "\Delta_{(1,\dots,1)\,*}"] \arrow[d, "\simeq"] & H^*_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl}) \arrow[d, "\simeq"] \\ H^*_{\mathcal{P}}(\Gamma^{d}\mathrm{gl}_{E_1}) \arrow[r, "\Delta_{(1,\dots,1)\,*}"] & H^*_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1}) \end{tikzcd} .$$

The image of the top horizontal arrow lives inside $H^*_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})^{\mathfrak{S}_d}$, the image of the bottom horizontal arrow lives inside $H^*_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1})^{\mathfrak{S}_d}$ and by naturality of the vertical arrow on the right, we have an isomorphism $H^*_{\mathcal{P}}(\otimes^{d\,(1)}\mathrm{gl})^{\mathfrak{S}_d} \simeq H^*_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1})^{\mathfrak{S}_d}$. Thus, to prove proposition 4.3, it suffices to prove that all the elements of $H^*_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1})^{\mathfrak{S}_d}$ are hit by the bottom horizontal arrow.

But $\otimes^d\mathrm{gl}$ (hence the summands of $\otimes^d\mathrm{gl}_{E_1}$) is injective in $\mathcal{P}^d_d$, whence an equality

$$H^*_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1}) = H^0_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1}) .$$

Now the left exactness of the functor $B \mapsto H^0_{\mathcal{P}}(B_{E_1})$ implies that the map

$$\Delta_{(1,\dots,1)\,*} : H^0_{\mathcal{P}}(\Gamma^d\mathrm{gl}_{E_1}) \to H^0_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1})^{\mathfrak{S}_d} = H^*_{\mathcal{P}}(\otimes^{d}\mathrm{gl}_{E_1})^{\mathfrak{S}_d}$$

is surjective. This concludes the proof. ☐

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